Differential Geometry of Some

SURFACES IN 3-SPACE

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December 2015

Introduction. Recent correspondence with Ahmed Sebbar concerning the theory of unimodular 3×3 circulant matrices[1]

$$\det\begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix} = x^3 + y^3 + z^3 - 3xyz = 1$$

brought to my attention a surface Σ in $\mathbb{R}^3$ which, I was informed, is encountered in work of H. Jonas (1915, 1921) and, because of its form when plotted, is known as "Jonas' hexenhut" (witch's hat). I was led by Google from "hexenhut" to a monograph *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory*, by C. Rogers & W. K. Schief (2002). These are subjects in which I have had longstanding interest, but which I have not thought about for many years. I am inspired by those authors' splendid book to revisit this subject area. In PART ONE I assemble the tools that play essential roles in the theory of surfaces in $\mathbb{R}^3$, and in PART TWO use those tools to develop the properties of some specific surfaces—particularly the pseudosphere, because it was the cradle in which was born the sine-Gordon equation, which a century later became central to the physical theory of solitons.[2]

PART ONE

Concepts & Tools Essential to the Theory of Surfaces in 3-Space

Fundamental forms. Relative to a Cartesian frame in $\mathbb{R}^3$, surfaces Σ can be described implicitly

$$f(x, y, z) = 0$$

but for the purposes of differential geometry must be described parametrically

$$\boldsymbol{r}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

[1] See "Simplest generalization of Pell's Problem," (September, 2015).
[2] That story is summarized in a companion essay, "Some remarks concerning the sine-Gordon equation," (November, 2015).

Tangent to Σ at the (non-singular) generic point $\boldsymbol{r}$ are the vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$, which are, in general, neither normalized nor orthogonal, but which will be assumed to be not parallel.

The local metric structure of Σ is indicated by

$$\begin{aligned} ds^2 = d\boldsymbol{r}\cdot d\boldsymbol{r} &= (\boldsymbol{r}_u du + \boldsymbol{r}_v dv)\cdot(\boldsymbol{r}_u du + \boldsymbol{r}_v dv) \\ &= \boldsymbol{r}_u\cdot\boldsymbol{r}_u dudu + 2\boldsymbol{r}_u\cdot\boldsymbol{r}_v dudv + \boldsymbol{r}_v\cdot\boldsymbol{r}_v dvdv \\ &\equiv E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2 \\ &= g_{ij}(u,v)du^i du^j \quad : \quad \text{here } u^1 = u, u^2 = v \end{aligned} \tag{1}$$

where

$$\mathbb{G}(u,v) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is the familiar metric tensor. Equation (1) is known as the 1st FUNDAMENTAL FORM.

The unit vector

$$\boldsymbol{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{r_u r_v \sin\omega}$$

is normal to both $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$, and therefore stands normal to Σ at $\boldsymbol{r}$. Differential variation of $\boldsymbol{N}$

$$d\boldsymbol{N} = \boldsymbol{N}_u du + \boldsymbol{N}_v dv$$

provides indication of the local curvature of Σ. That information is folded into the structure of the 2nd FUNDAMENTAL FORM

$$-d\boldsymbol{r}\cdot d\boldsymbol{N} = -\big\{\boldsymbol{r}_u\cdot\boldsymbol{N}_u dudu + (\boldsymbol{r}_u\cdot\boldsymbol{N}_v + \boldsymbol{r}_v\cdot\boldsymbol{N}_u)dudv + \boldsymbol{r}_v\cdot\boldsymbol{N}_v dvdv\big\}$$

which by arguments of the form

$$(\boldsymbol{r}_p\cdot\boldsymbol{N})_q = 0_q = \boldsymbol{r}_p\cdot\boldsymbol{N}_q + \boldsymbol{r}_{pq}\cdot\boldsymbol{N} \quad\Longrightarrow\quad -\boldsymbol{r}_p\cdot\boldsymbol{N}_q = \boldsymbol{r}_{pq}\cdot\boldsymbol{N}$$

can be written

$$\begin{aligned} -d\boldsymbol{r}\cdot d\boldsymbol{N} &= \boldsymbol{r}_{uu}\cdot\boldsymbol{N}du^2 + 2\boldsymbol{r}_{uv}\cdot\boldsymbol{N}dudv + \boldsymbol{r}_{vv}\cdot\boldsymbol{N}dv^2 \\ &\equiv e(u,v)du^2 + 2f(u,v)dudv + g(u,v)dv^2 \\ &= h_{ij}(u,v)du^i du^j \end{aligned} \tag{2}$$

with

$$\mathbb{H}(u,v) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

From $ds^2 > 0$ it follows that the metric matrix $\mathbb{G}$ must be positive-definite (both eigenvalues positive, their product $\det\mathbb{G} > 0$). No such condition pertains, however, to $\mathbb{H}$, the determinant of which can be positive at some points P on Σ, vanish or be negative at other points. We shall, in fact, have special interest in "hyperbolic" surfaces, on which at all points $\det\mathbb{H} < 0$.[3]

[3] It is unfortunate that tradition has assigned roles to the symbols G and g to which the conventions of Riemannian geometry have assigned other roles; we are thus precluded from writing (for example) $g = \det\mathbb{G}$. I will depart only occasionally from tradition, and endeavor always to minimize the possibility of confusion.

$\mathbb{G}$ conveys information that is locally *intrinsic* to the surface. Not so $\mathbb{H}$, which by its $\boldsymbol{N}$-dependence is rendered *extrinsic*.

The Gauss equations. These present $\boldsymbol{r}_{uu}$, $\boldsymbol{r}_{uv}$ and $\boldsymbol{r}_{vv}$ as linear combinations of $\{\boldsymbol{r}_u, \boldsymbol{r}_v, \boldsymbol{N}\}$. They can be produced as corollaries of the following

LEMMA: Let $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$—linearly independent, but subject to no additional restrictions—comprise an arbitrary basis in 3-space, and let the arbitrary vector $\boldsymbol{x}$ be developed

$$\boldsymbol{x} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$$

Then

$$\begin{pmatrix} \boldsymbol{a\cdot a} & \boldsymbol{a\cdot b} & \boldsymbol{a\cdot c} \\ \boldsymbol{b\cdot a} & \boldsymbol{b\cdot b} & \boldsymbol{b\cdot c} \\ \boldsymbol{c\cdot a} & \boldsymbol{c\cdot b} & \boldsymbol{c\cdot c} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \boldsymbol{a\cdot x} \\ \boldsymbol{b\cdot x} \\ \boldsymbol{c\cdot x} \end{pmatrix}$$

which if $\boldsymbol{c}$ is a unit vector normal to both $\boldsymbol{a}$ and $\boldsymbol{b}$ becomes

$$\begin{pmatrix} \boldsymbol{a\cdot a} & \boldsymbol{a\cdot b} & 0 \\ \boldsymbol{b\cdot a} & \boldsymbol{b\cdot b} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \boldsymbol{a\cdot x} \\ \boldsymbol{b\cdot x} \\ \boldsymbol{c\cdot x} \end{pmatrix}$$

giving

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{1}{D}\begin{pmatrix} \boldsymbol{b\cdot b} & -\boldsymbol{a\cdot b} & 0 \\ -\boldsymbol{a\cdot b} & \boldsymbol{a\cdot a} & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} \boldsymbol{a\cdot x} \\ \boldsymbol{b\cdot x} \\ \boldsymbol{c\cdot x} \end{pmatrix}$$

where

$$D = (\boldsymbol{a\cdot a})(\boldsymbol{b\cdot b}) - (\boldsymbol{a\cdot b})^2 = \det\begin{pmatrix} \boldsymbol{a\cdot a} & \boldsymbol{a\cdot b} \\ \boldsymbol{a\cdot b} & \boldsymbol{b\cdot b} \end{pmatrix}$$

We therefore have

$$\begin{aligned} \boldsymbol{x} = {} & D^{-1}\big[\,(\boldsymbol{b\cdot b})(\boldsymbol{a\cdot x}) - (\boldsymbol{a\cdot b})(\boldsymbol{b\cdot x})\big]\,\boldsymbol{a} \\ & + D^{-1}\big[(\boldsymbol{a\cdot a})(\boldsymbol{b\cdot x}) - (\boldsymbol{a\cdot b})(\boldsymbol{a\cdot x})\big]\,\boldsymbol{b} + (\boldsymbol{c\cdot x})\,\boldsymbol{c} \end{aligned} \tag{3}$$

Now set $\boldsymbol{a} \to \boldsymbol{r}_u, \boldsymbol{b} \to \boldsymbol{r}_v, \boldsymbol{c} \to \boldsymbol{N}$ and look to the case $\boldsymbol{x} \to \boldsymbol{r}_{uu}$. We then have

$$D = \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \text{ which at risk of confusion will be denoted } g$$

$$\begin{aligned} \boldsymbol{r}_{uu} = g^{-1}\big[g_{22}(\boldsymbol{r}_u\boldsymbol{\cdot r}_{uu}) - g_{12}(\boldsymbol{r}_v\boldsymbol{\cdot r}_{uu})\big]\boldsymbol{r}_u + g^{-1}\big[g_{11}(\boldsymbol{r}_v\boldsymbol{\cdot r}_{uu}) - g_{12}(\boldsymbol{r}_u\boldsymbol{\cdot r}_{uu})\big]\boldsymbol{r}_v \\ + (\boldsymbol{N\cdot r}_{uu})\,\boldsymbol{N} \end{aligned}$$

which by

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = g^{-1}\begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix}$$

becomes

$$\boldsymbol{r}_{uu} = \big[g^{11}(\boldsymbol{r}_u\boldsymbol{\cdot r}_{uu}) + g^{12}(\boldsymbol{r}_v\boldsymbol{\cdot r}_{uu})\big]\,\boldsymbol{r}_u + \big[g^{22}(\boldsymbol{r}_v\boldsymbol{\cdot r}_{uu}) + g^{21}(\boldsymbol{r}_u\boldsymbol{\cdot r}_{uu})\big]\,\boldsymbol{r}_v + e\boldsymbol{N}$$

But

$$\begin{aligned}(\boldsymbol{r}_u \cdot \boldsymbol{r}_{uu}) &= \tfrac{1}{2}(\boldsymbol{r}_u \cdot \boldsymbol{r}_u)_u = \tfrac{1}{2}g_{11,1}\\(\boldsymbol{r}_v \cdot \boldsymbol{r}_{uu}) &= (\boldsymbol{r}_u \cdot \boldsymbol{r}_v)_u - \tfrac{1}{2}(\boldsymbol{r}_u \cdot \boldsymbol{r}_u)_v = g_{12,1} - \tfrac{1}{2}g_{11,2}\end{aligned}$$

$$\begin{aligned}(\boldsymbol{r}_u \cdot \boldsymbol{r}_{uv}) &= \tfrac{1}{2}(\boldsymbol{r}_u \cdot \boldsymbol{r}_u)_v = \tfrac{1}{2}g_{11,2}\\(\boldsymbol{r}_v \cdot \boldsymbol{r}_{uv}) &= \tfrac{1}{2}(\boldsymbol{r}_v \cdot \boldsymbol{r}_v)_u = \tfrac{1}{2}g_{22,1}\end{aligned}$$

$$\begin{aligned}(\boldsymbol{r}_u \cdot \boldsymbol{r}_{vv}) &= (\boldsymbol{r}_u \cdot \boldsymbol{r}_v)_v - \tfrac{1}{2}(\boldsymbol{r}_v \cdot \boldsymbol{r}_v)_u = g_{12,2} - \tfrac{1}{2}g_{22,1}\\(\boldsymbol{r}_v \cdot \boldsymbol{r}_{vv}) &= \tfrac{1}{2}(\boldsymbol{r}_v \cdot \boldsymbol{r}_v)_v = \tfrac{1}{2}g_{22,2}\end{aligned}$$

so we have

$$\begin{aligned}\boldsymbol{r}_{uu} = &\quad \big[\tfrac{1}{2}g^{11}g_{11,1} + g^{12}g_{12,1} - \tfrac{1}{2}g^{12}g_{11,2}\big]\boldsymbol{r}_u\\&+ \big[\tfrac{1}{2}g^{21}g_{11,1} + g^{22}g_{12,1} - \tfrac{1}{2}g^{22}g_{11,2}\big]\boldsymbol{r}_v + e\boldsymbol{N}\end{aligned}$$

From the definition of the Christoffel symbols

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{im}\big(g_{jm,k} + g_{km,j} - g_{jk,m}\big)$$

we have

$$\begin{aligned}\Gamma^1_{11} &= \tfrac{1}{2}g^{11}g_{11,1} + g^{12}g_{12,1} - \tfrac{1}{2}g^{12}g_{11,2}\\\Gamma^2_{11} &= \tfrac{1}{2}g^{21}g_{11,1} + g^{22}g_{12,1} - \tfrac{1}{2}g^{22}g_{11,2}\end{aligned}$$

so by this and similar arguments we arrive finally at the GAUSS EQUATIONS[4]

$$\left.\begin{aligned}\boldsymbol{r}_{11} &= \Gamma^1_{11}\boldsymbol{r}_1 + \Gamma^2_{11}\boldsymbol{r}_2 + e\boldsymbol{N}\\\boldsymbol{r}_{12} &= \Gamma^1_{12}\boldsymbol{r}_1 + \Gamma^2_{12}\boldsymbol{r}_2 + f\boldsymbol{N}\\\boldsymbol{r}_{22} &= \Gamma^1_{22}\boldsymbol{r}_1 + \Gamma^2_{22}\boldsymbol{r}_2 + g\boldsymbol{N}\end{aligned}\right\} \tag{4.1}$$

The subscripted notation tends to obscure what has here been accomplished; it is perhaps more vividly informative to write

$$\left.\begin{aligned}\boldsymbol{r}_{uu} &= \Gamma^1_{11}\boldsymbol{r}_u + \Gamma^2_{11}\boldsymbol{r}_v + e\boldsymbol{N}\\\boldsymbol{r}_{uv} &= \Gamma^1_{12}\boldsymbol{r}_u + \Gamma^2_{12}\boldsymbol{r}_v + f\boldsymbol{N}\\\boldsymbol{r}_{vv} &= \Gamma^1_{22}\boldsymbol{r}_u + \Gamma^2_{22}\boldsymbol{r}_v + g\boldsymbol{N}\end{aligned}\right\} \tag{4.2}$$

The Gauss equations are linear, and—because they involve $\boldsymbol{N}$—extrinsic.

In the next section we proceed from the LEMMA, by a similar argument, to a pair of equations that were introduced in 1861 by J. Weingarten (1836–1910).

[4] The occurance of Christoffel symbols in these equations is a bit of an anachronism, since it was mainly during the years 1820–1830 that Gauss (1777–1855) concerned himself with differential geometry, and he had been dead for fourteen years by the time Elwin Christoffel (1829–1900) introduced the symbols that bear his name.

The Weingarten equations. These develop $\boldsymbol{N}_u$ and $\boldsymbol{N}_v$ as linear combinations ofe $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$. We expect such formulae to exist, since the unit vector $\boldsymbol{N}$ stands normal to Σ at $\boldsymbol{r}$, and (by differentiation of $\boldsymbol{N}\cdot\boldsymbol{N}=1$) $\boldsymbol{N}_u\cdot\boldsymbol{N}=\boldsymbol{N}_v\cdot\boldsymbol{N}=0$ so $\boldsymbol{N}_u$ and $\boldsymbol{N}_v$ lie in the plane tangent to Σ at $\boldsymbol{r}$, which is spanned by $\{\boldsymbol{r}_u,\boldsymbol{r}_v\}$. To construct such formulae we return to (3), set $\boldsymbol{a}\to\boldsymbol{r}_u, \boldsymbol{b}\to\boldsymbol{r}_v, \boldsymbol{c}\to\boldsymbol{N}$ as before but this time look to the cases $\boldsymbol{x}\to\boldsymbol{N}_u$ else $\boldsymbol{N}_v$. We then have

$$\begin{aligned}\boldsymbol{N}_u &= g^{-1}\big[g_{22}(\boldsymbol{r}_u\cdot\boldsymbol{N}_u)-g_{12}(\boldsymbol{r}_v\cdot\boldsymbol{N}_u)\big]\boldsymbol{r}_u+g^{-1}\big[g_{11}(\boldsymbol{r}_v\cdot\boldsymbol{N}_u)-g_{12}(\boldsymbol{r}_u\cdot\boldsymbol{N}_u)\big]\boldsymbol{r}_v\\ &\qquad +(\boldsymbol{N}\cdot\boldsymbol{N}_u)\boldsymbol{N}\\ \boldsymbol{N}_v &= g^{-1}\big[g_{22}(\boldsymbol{r}_u\cdot\boldsymbol{N}_v)-g_{12}(\boldsymbol{r}_v\cdot\boldsymbol{N}_v)\big]\boldsymbol{r}_u+g^{-1}\big[g_{11}(\boldsymbol{r}_v\cdot\boldsymbol{N}_v)-g_{12}(\boldsymbol{r}_u\cdot\boldsymbol{N}_v)\big]\boldsymbol{r}_v\\ &\qquad +(\boldsymbol{N}\cdot\boldsymbol{N}_v)\boldsymbol{N}\end{aligned}$$

which by

$$\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}=\begin{pmatrix} E & F\\ F & G\end{pmatrix},\qquad \begin{pmatrix} \boldsymbol{r}_u\cdot\boldsymbol{N}_u & \boldsymbol{r}_u\cdot\boldsymbol{N}_v\\ \boldsymbol{r}_v\cdot\boldsymbol{N}_u & \boldsymbol{r}_v\cdot\boldsymbol{N}_v\end{pmatrix}=\begin{pmatrix} -e & -f\\ -f & -g\end{pmatrix}$$

$g^{-1}=(EG-F^2)^{-1}$ and $\boldsymbol{N}_u\cdot\boldsymbol{N}=\boldsymbol{N}_v\cdot\boldsymbol{N}=0$ give

$$\left.\begin{aligned}\boldsymbol{N}_u &= \frac{fF-eG}{EG-F^2}\boldsymbol{r}_u+\frac{eF-fE}{EG-F^2}\boldsymbol{r}_v\\ \boldsymbol{N}_v &= \frac{gF-fG}{EG-F^2}\boldsymbol{r}_u+\frac{fF-gE}{EG-F^2}\boldsymbol{r}_v\end{aligned}\right\}\tag{5}$$

which are the (extrinsic linear) WEINGARTEN EQUATIONS.

The Mainardi-Codazzi equations. From the requirement that the first pair of Gauss equations (4.2) conform, as a "compatability condition," to the identity $(\boldsymbol{r}_{uu})_v=(\boldsymbol{r}_{uv})_u$ we have

$$\begin{aligned}&(\Gamma^1_{11})_v\boldsymbol{r}_u+\Gamma^1_{11}\boldsymbol{r}_{uv}+(\Gamma^2_{11})_v\boldsymbol{r}_v+\Gamma^2_{11}\boldsymbol{r}_{vv}+e_v\boldsymbol{N}+e\boldsymbol{N}_v\\ &=(\Gamma^1_{12})_u\boldsymbol{r}_u+\Gamma^1_{12}\boldsymbol{r}_{uu}+(\Gamma^2_{12})_u\boldsymbol{r}_v+\Gamma^2_{12}\boldsymbol{r}_{uv}+f_u\boldsymbol{N}+f\boldsymbol{N}_u\end{aligned}$$

which when dotted into $\boldsymbol{N}$ becomes

$$\Gamma^1_{11}f+\Gamma^2_{11}g+e_v=\Gamma^1_{12}e+\Gamma^2_{12}f+f_u$$

or

$$\left.\begin{aligned}e_v-f_u &= e\Gamma^1_{12}+f(\Gamma^2_{12}-\Gamma^1_{11})-g\Gamma^2_{11}\\ f_v-g_u &= e\Gamma^1_{22}+f(\Gamma^2_{22}-\Gamma^1_{12})-g\Gamma^2_{12}\end{aligned}\right\}\tag{6}$$

where the second of these MAINARDI-CODAZZI EQUATIONS was derived by a similar argument from the second pair of Gauss equations by $(\boldsymbol{r}_{uv})_v=(\boldsymbol{r}_{vv})_u$. These equations, sometimes called "Gauss-Codazzi equations," were discovered in 1856 by Gaspare Mainardi (1800–1879), and independently in 1868 by Delfino Codazzi (1824–1873), but first appear in the dissertation (1853) of Karl Peterson.

The Mainardi-Codazzi equations—corollaries by a compatability condition of the Gauss equations—describe a fundamental relation between the components of the 2^{nd} fundamental form and their derivatives. Pierre Bonnet (1819–1892) showed that surfaces that satisfy the same Mainardi-Codazzi equations can be made to coincide by a Euclidean transformation (translation + rotation). The equations are therefore central to the theory of embedded surfaces.

The shape operator. The Weingarten equations (5), which describe a linear transformation ("Weingarten map")

$$\{\boldsymbol{r}_u, \boldsymbol{r}_v\} \to \{\boldsymbol{N}_u, \boldsymbol{N}_v\}$$

on the tangent plane at $\boldsymbol{r}$, can be written

$$\begin{aligned}\begin{pmatrix}\boldsymbol{N}_u\\ \boldsymbol{N}_v\end{pmatrix} &= -\frac{1}{EG-F^2}\begin{pmatrix} eG-fF & fE-eF\\ fG-gF & gE-fF\end{pmatrix}\begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\end{pmatrix}\\ &\equiv -\mathbb{S}\begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\end{pmatrix}\end{aligned} \tag{7}$$

where $\mathbb{S}$—assembled from the components of the 1^{st} and 2^{nd} fundamental forms —is the SHAPE OPERATOR, an object introduced by the (Nazi) mathematician Wilhelm Blaschke (1885–1962) and not mentioned by Rogers & Schief, but to which an entire chapter is devoted in Barrett O'Neill's *Elementary Differential Geometry* (1966, revised 2^{nd} edition 2006), which I learned about from the "Differential geometry of surfaces" article in Wikipedia and which is now avaiable on the web as a free download.

The shape operator gains interest partly from the fact that its eigenvalues are the principal curvatures $\{\kappa_1, \kappa_2\}$ of Σ at $\boldsymbol{r}$; *Mathematica* supplies

$$\begin{aligned}\det\mathbb{S} = \frac{eg-f^2}{EG-F^2} &= \begin{cases}\text{product of eigenvalues}\\ \text{product of principal curvatures}\end{cases}\\ &= \text{Gaussian curvature } K \equiv \kappa_1\!\cdot\kappa_2\end{aligned} \tag{8.1}$$

$$\begin{aligned}\tfrac{1}{2}\mathrm{tr}\,\mathbb{S} = \frac{gE-2fF+eG}{EG-F^2} &= \begin{cases}\text{average of eigenvalues}\\ \text{average of principal curvatures}\end{cases}\\ &= \text{mean curvature } K_m = \tfrac{1}{2}(\kappa_1+\kappa_2)\end{aligned} \tag{8.2}$$

Inversely

$$\left.\begin{matrix}\kappa_1\\ \kappa_2\end{matrix}\right\} = K_m \pm \sqrt{K_m^2 - K} \tag{9}$$

Curvature. At (8) I imported a pair of Gauss' classic results. The theory of the curvature of curves and surfaces is a sprawling subject that can be approached in a great many ways and with various degrees of abstract formality. My limited intent in the informal remarks which follow will be to assemble (and to attempt to place in context) only the results of which I will have specific need.

To begin at the beginning: Look to the graph of $y(x)$, viewed as a curve $\mathcal{C}$ on the Euclidean plane $\mathbb{R}^2$. From F. L. Griffin I learned more than six decades ago that the curvature of $\mathcal{C}$ at x is defined[5]

$$\begin{aligned}\kappa(x) &= \text{derivative of local slope with respect to arc length}\\ &= \tfrac{d\tau}{ds} \quad : \quad \tau = \text{slope} = \arctan \tfrac{dy}{dx}\\ &= \tfrac{d\tau}{dx}\cdot\tfrac{dx}{ds}\\ &= \big[1+\big(\tfrac{dy}{dx}\big)^2\big]^{-1}\tfrac{d^2y}{dx^2}\cdot\big(\tfrac{ds}{dx}\big)^{-1}\\ &= \frac{\tfrac{d^2y}{dx^2}}{\big[1+\big(\tfrac{dy}{dx}\big)^2\big]^{\frac{3}{2}}} \quad \text{by} \quad ds = \big[1+\big(\tfrac{dy}{dx}\big)^2\big]^{\frac{1}{2}}dx \end{aligned}\tag{9.1}$$

Suppose now that $\mathcal{C}$ has been described parametrically:

$$\boldsymbol{r}(u) = \begin{pmatrix} x(u)\\ y(u)\end{pmatrix}$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{y'}{x'} \quad \text{where } ' \text{ signifies differentiation with respect to } u\\ \frac{d^2y}{dx^2} &= \frac{du}{dx}\cdot\frac{d}{du}(y'/x')\\ &= \frac{1}{x'}\cdot\frac{x'y''-y'x''}{x'x'}\end{aligned}$$

which by (9.1) give

$$\kappa(u) = \frac{x'y''-y'x''}{[x'^2+y'^2]^{\frac{3}{2}}} = \frac{1}{[x'^2+y'^2]^{\frac{3}{2}}}\det\begin{pmatrix} x' & x''\\ y' & y''\end{pmatrix}\tag{9.2}$$

Introduce

$$\begin{aligned}\boldsymbol{t} &= \frac{1}{[x'^2+y'^2]^{\frac{1}{2}}}\boldsymbol{r}' &&: \quad \text{unit tangent vector at } \boldsymbol{r}\\ \boldsymbol{n} &= \frac{1}{[x'^2+y'^2]^{\frac{1}{2}}}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\boldsymbol{r}' &&: \quad \text{unit vector} \perp \boldsymbol{t}\end{aligned}$$

We then have $\boldsymbol{r}'' = \alpha\boldsymbol{t} + \beta\boldsymbol{n}$ with

$$\begin{aligned}\alpha = \boldsymbol{t}\cdot\boldsymbol{r}'' &= \frac{x'x''+y'y''}{[x'^2+y'^2]^{\frac{1}{2}}}\\ \beta = \boldsymbol{n}\cdot\boldsymbol{r}'' &= \frac{x'y''-y'x''}{[x'^2+y'^2]^{\frac{1}{2}}}\end{aligned}$$

so by (9.2)

$$\kappa(u) = \frac{1}{[x'^2+y'^2]}\beta = \frac{1}{[x'^2+y'^2]}(\text{magnitude of the } \boldsymbol{n}\text{-component of } \boldsymbol{r}'')$$

[5] *Mathematical Analysis: Higher Course* (1927), page 422.

Argued another way, we by $\boldsymbol{t}\cdot\boldsymbol{t} = 1$ have $\boldsymbol{t}\cdot\boldsymbol{t}' = 0 \implies \boldsymbol{t}' \sim \boldsymbol{n}$. And by quick calculation

$$\begin{aligned}
\boldsymbol{t}' &= \frac{1}{[x'^2 + y'^2]^{\frac{3}{2}}}\left[-(x'x'' + y'y'')\begin{pmatrix} x' \\ y' \end{pmatrix} + (x'^2 + y'^2)\begin{pmatrix} x'' \\ y'' \end{pmatrix}\right] \\
&= \frac{x'y'' - y'x''}{[x'^2 + y'^2]}\left[\frac{1}{[x'^2 + y'^2]^{\frac{1}{2}}}\right]\begin{pmatrix} -y' \\ +x' \end{pmatrix} \\
&= \frac{x'y'' - y'x''}{[x'^2 + y'^2]}\boldsymbol{n} \\
&= \sqrt{x'^2 + y'^2}\,\kappa(u)\,\boldsymbol{n}
\end{aligned}$$

If the parameter u is taken to be arc length s then by $ds^2 = dx^2 + dy^2$ we have $\sqrt{x'^2 + y'^2} = 1$ and the preceding equation assumes the simple form

$$\tfrac{d}{ds}\boldsymbol{t}(s) = \kappa(s)\,\boldsymbol{n}(s) = \kappa(s)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\boldsymbol{t} \tag{9.3}$$

Equations (9.1) and (9.2) describe a local property of $\mathcal{C}$ in terms that refer extrinsically to $\boldsymbol{r}$. Equation (9.3) is, on the other hand, intrinsic; given $\{\boldsymbol{r}(0), \boldsymbol{t}(0)\}$ and $\kappa(s)$ one could reconstruct $\mathcal{C}$. Adjusting the values of $\{\boldsymbol{r}(0), \boldsymbol{t}(0)\}$ —while preserving the structure of $\kappa(s)$—would lead to a different curve $\mathcal{C}'$ which is congruent to but not coincident with $\mathcal{C}$, *i.e.*, which is a Euclidean transform of $\mathcal{C}$.

Turning now from plane curves to space curves, let $\boldsymbol{r}(s)$ refer to a curve $\mathcal{C}$ in $\mathbb{R}^3$. Then

$$\boldsymbol{t}(s) = \tfrac{d}{ds}\boldsymbol{r}(s) = \text{unit tangent to } \mathcal{C} \text{ at } s$$

and by differentation of $\boldsymbol{t}(s)\cdot\boldsymbol{t}(s) = 1$ we have

$$\tfrac{d}{ds}\boldsymbol{t}(s) = \kappa(s)\,\boldsymbol{u}(s)$$

where $\boldsymbol{u}(s)$ is a unit vector normal to $\boldsymbol{t}(s)$:

$$\boldsymbol{u}(s)\cdot\boldsymbol{t}(s) = 0 \quad\text{and}\quad \boldsymbol{u}(s)\cdot\boldsymbol{u}(s) = 1$$

$\boldsymbol{u}(s)$ describes the direction, and $\kappa(s)$ the magnitude, of the local curvature of $\mathcal{C}$. Assume $\kappa(s) \neq 0$ and define $\boldsymbol{v}(s) \equiv \boldsymbol{t}(s) \times \boldsymbol{u}(s)$, which serves to complete the construction of an orthonormal triad at each (non-straight) point s of $\mathcal{C}$. Elementary arguments[6] lead to the conclusions that

$$\tfrac{d}{ds}\boldsymbol{u}(s) = -\kappa(s)\,\boldsymbol{t}(s) - \tau(s)\,\boldsymbol{v}(s) \quad\text{and}\quad \tfrac{d}{ds}\boldsymbol{v}(s) = \tau(s)\,\boldsymbol{u}(s)$$

where $\tau(s)$ is the torsion of $\mathcal{C}$ at s. We arrive thus at

[6] See my "Frenet-Serret formulae in higher dimension" (August 1998) and the essay cited there.

$$\frac{d}{ds}\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} \tag{10}$$

which comprise the famous FRENET-SERRET FORMULAE.[7] They serve in effect to describe the rotation matrix $\mathbb{R}(s)$ that relates the $\{\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}\}$-frame at s to the frame at $s + ds$.

If $\boldsymbol{r}(u, v)$ describes a surface Σ in $\mathbb{R}^3$ then $\boldsymbol{r}(u(s), v(s))$ describes a curve $\mathcal{C}$ inscribed on Σ. Thus does the theory of space curves acquire direct relevance to the curvature theory of surfaces. Look in particular to the curves inscribed on a neighborhood surrounding the (u, v)-point P on Σ by its intersection with the planes that contain $\boldsymbol{N}(u, v)$, the local normal to Σ. For such an inscribed curve we have[8]

$$\boldsymbol{t} = \boldsymbol{r}_s = \boldsymbol{r}_u u_s + \boldsymbol{r}_v v_s$$

and therefore

$$\boldsymbol{t}_s = \boldsymbol{r}_{uu} u_s^2 + 2\,\boldsymbol{r}_{uv} u_s v_s + \boldsymbol{r}_{vv} v_s^2 + \boldsymbol{r}_u u_{ss} + \boldsymbol{r}_v v_{ss} = \kappa_n \boldsymbol{N} \tag{11}$$

Dot $\boldsymbol{N}$ into this result. By $\boldsymbol{N}\cdot\boldsymbol{r}_u = \boldsymbol{N}\cdot\boldsymbol{r}_v = 0$, $\boldsymbol{N}\cdot\boldsymbol{N} = 1$ and recalling from (2) the definitions of the coefficients in the 2nd fundamental form obtain

$$\begin{aligned} \kappa_n &= (\boldsymbol{r}_{uu}\cdot\boldsymbol{N})u_s^2 + 2\,(\boldsymbol{r}_{uv}\cdot\boldsymbol{N})u_s v_s + (\boldsymbol{r}_{vv}\cdot\boldsymbol{N})v_s^2 \\ &= e(u,v)u_s^2 + 2f(u,v)u_s v_s + g(u,v)v_s^2 \\ &= \frac{e\,dudu + 2f\,dudv + g\,dvdv}{ds^2} \\ &= \frac{e\,dudu + 2f\,dudv + g\,dvdv}{E\,dudu + 2F\,dudv + G\,dvdv} \\ &= \text{local curvature of such a “normal curve” } \mathcal{C}_n \end{aligned} \tag{12}$$

I digress to remark that for an *arbitrary* curve $\mathcal{C}$ passing through P with the same $\boldsymbol{t}$ the values of $\boldsymbol{r}_{uu} u_s^2 + 2\,\boldsymbol{r}_{uv} u_s v_s + \boldsymbol{r}_{vv} v_s^2$, $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ remain the same as they were for $\mathcal{C}_n$; it is owing entirely to adjustments in the values of the second derivatives u_{ss} and v_{ss} that (11) yields $\kappa\boldsymbol{u}$ instead of $\kappa_n\boldsymbol{N}$. Which is to say, (11) assumes the forms

$$\begin{aligned} \boldsymbol{F} + \boldsymbol{r}_u u_{ss} + \boldsymbol{r}_v v_{ss} &= \kappa_n \boldsymbol{N} \\ \boldsymbol{F} + \boldsymbol{r}_u u'_{ss} + \boldsymbol{r}_v v'_{ss} &= \kappa\,\boldsymbol{u} \end{aligned}$$

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[7] These, when written out in component form (as they necessarily were prior to the invention of vector algebra), are a set of nine equations, of which Jean Frenet (1816–1900) discovered six in 1847 and Joseph Serret (1819–1885) the remaining three in 1851. I would have expected these discoveries to have occurred much earlier in the history of differential geometry, and to have been known in particular to Gauss.

[8] Here I follow Harry Lass, *Vector and Tensor Analysis* (1950), pages 74–78, whose discussion I admire not for its formal elegance but for its conceptual clarity and analytical swiftness.

for curves of types $\mathcal{C}_n$ and $\mathcal{C}$, respectively. Dotting $\boldsymbol{N}$ into those equations, and noting that $\boldsymbol{N}$ and $\boldsymbol{u}$ are both unit vectors (normal to $\boldsymbol{t}$), we obtain

$$\boldsymbol{N}\cdot\boldsymbol{F} + 0 = \kappa_n$$
$$\boldsymbol{N}\cdot\boldsymbol{F} + 0 = \kappa\,\boldsymbol{N}\cdot\boldsymbol{u} = \kappa\cos\theta$$

whence

$$\kappa = \kappa_n \sec\theta \tag{13}$$

which is MEUSNIER'S THEOREM, a result discovered by Jean Baptiste Meusnier (1754–1793) in 1776, very early in the history of differential geometry.

From (12) we have

$$(\kappa_n E - e)du^2 + 2(\kappa_n F - f)dudv + (\kappa_n G - g)dv^2 = 0 \tag{14.1}$$

which written

$$A(\kappa_n)(\tfrac{du}{dv})^2 + 2B(\kappa_n)\tfrac{du}{dv} + C(\kappa_n) = 0$$

assigns—for every given value of κ_n—two possible values to $\frac{du}{dv}$:

$$\Big(\frac{du}{dv}\Big)_\pm = \frac{-B \pm \sqrt{B^2 - AC}}{A} \tag{14.2}$$

Twirl the normal plane about its $\boldsymbol{N}$-axis, inscribing on Σ all possible $\mathcal{C}_n$-curves that pass through P. Their normal curvatures will range on $\{\kappa_{n,min}, \kappa_{n,max}\}$ and will assume their extremal values when the expressions on the right side of (14.2) are coincident; *i.e.*, when

$$\begin{aligned} B^2 - AC &= (\kappa_n F - f)^2 - (\kappa_n E - e)(\kappa_n G - g) \\ &= (F^2 - EG)\kappa_n^2 + (Eg - 2Ff + Ge)\kappa_n + (f^2 - eg) \\ &\equiv a\kappa_n^2 + b\kappa_n + c = 0 \end{aligned}$$

The roots of this equation are

$$\kappa_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$\kappa_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

which are the "principal curvatures," the Frenet-Serret curvatures of the "principal directions" at the point P on Σ. The GAUSSIAN CURVATURE at P is, by definition,

$$K = \kappa_1\kappa_2 = \frac{c}{a} = \frac{eg - f^2}{EG - F^2} \tag{15.1}$$

and the MEAN CURVATURE is

$$K_m = \tfrac{1}{2}(\kappa_1 + \kappa_2) = -\frac{b}{2a} = \frac{Eg - 2Ff + Ge}{EG - F^2} \tag{15.2}$$

as was asserted already at (8).

The derivatives $(\frac{du}{dv})_\pm$ described by (14.2) become coincident if

$$\begin{aligned} B^2 - AC &= (\kappa_n F - f)^2 - (\kappa_n E - e)(\kappa_n G - g) \\ &= (F^2 - EG)\kappa_n^2 + (Eg - 2Ff + Ge)\kappa_n + (f^2 - eg) = 0 \end{aligned} \tag{16.1}$$

and are then given by

$$\frac{du}{dv} = -\frac{B}{A} = -\frac{\kappa_n F - f}{\kappa_n E - e} \tag{16.2}$$

From (16.1) it follows (non-obviously!) that

$$\begin{aligned} (Ef - Fe)(B^2 - AB) = \quad &(Ef - Fe)(\kappa_n F - f)^2 \\ &- (Eg - Ge)(\kappa_n F - f)(\kappa_n E - e) \\ &+ (Fg - Gf)(\kappa_n E - e)^2 = 0 \end{aligned}$$

which by (16.2), when divided by $(\kappa_n E - e)^2$, becomes (compare (14.1))

$$(Ef - Fe)\,du^2 + (Eg - Ge)\,dudv + (Fg - Gf)\,dv^2 = 0 \tag{16.3}$$

This is a single relation satisfied by both of the principal curves that pass through the point P on Σ. The coefficients in (16.3) are reminiscent of but distinct from those encountered as numerators in the Weingarten equations (5), and as elements of the shape operator (7).

To equation (15.1)—which can be written

$$K = \frac{\det \mathbb{H}}{\det \mathbb{G}}$$

—Gauss (1827) gave the name "*Theorema egregium*" (remarkable theorem) because the definition alludes to both fundamental forms (the 1st intrinsic, the 2nd extrinsic) yet describes an intrinsic property of surfaces in $\mathbb{R}^3$. Suppose some infinitely flexible but absolutely inextensible film to have been cast in the form Σ, on it to have been inscribed (arbitrarily) a $\{u, v\}$ coordinate system and the Gaussian curvature K at some designated point P to have been computed. Isometric deformations $\Sigma \to \Sigma'$ (arbitrary bending or pinching, no stretching) may—and typically will—alter the values of both κ_1 and κ_2 (and, indeed, also of their sum; *i.e.*, of the mean curvature at P) but preserve the value of their product $K = \kappa_1\kappa_2$.[9]

Because the Gaussian curvature K refers to an intrinsic local property of Σ we expect it to be describable in terms that refer only to manifestly intrinsic the local metric structure of the surface (1st fundamental form). The literature provides several such formulae. Here I must be content not to derive but simply to present a partial list of alternative metric constructions of K, and of some of their close relatives.

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[9] Think of a ruled cylindar, where isometric deformations may alter κ_1 (the cross section) but preserve $\kappa_2 = 0$. For an animated representation of an isometric transformation more complicated than "pinching," see "Theorema Egregium" in Wikipedia. This site provides links to Engish translations of Gauss' original papers.

Several of the formulae in question involve Christoffel symbols[10] of the first and second kinds:

$$\begin{aligned} \Gamma_{mjk} &= \qquad \tfrac{1}{2}\big(g_{mj,k} + g_{mk,j} - g_{jk,m}\big) = \Gamma_{mkj} \\ \Gamma^i{}_{jk} = g^{im}\Gamma_{mjk} &= \tfrac{1}{2}g^{im}\big(g_{mj,k} + g_{mk,j} - g_{jk,m}\big) = \Gamma^i{}_{kj} \end{aligned}$$

In the n-dimensional case these are (reduced by symmetry from n^3 to) $\frac{1}{2}n^2(n+1)$ in number, so for the 2-dimensional manifolds Σ of interest we confront a population of twelve Christoffel symbols (six of each kind), of which we will in fact need only the symbols of the second kind: in indexed notation

$$\left.\begin{aligned} \Gamma^1{}_{11} &= \tfrac{1}{2}g^{11}g_{11,1} + g^{12}g_{12,1} - \tfrac{1}{2}g^{12}g_{11,2} \\ \Gamma^2{}_{11} &= \tfrac{1}{2}g^{12}g_{11,1} + g^{22}g_{12,1} - \tfrac{1}{2}g^{22}g_{11,2} \\ \Gamma^1{}_{12} = \Gamma^1{}_{21} &= \tfrac{1}{2}g^{11}g_{11,2} + \tfrac{1}{2}g^{12}g_{22,1} \\ \Gamma^2{}_{12} = \Gamma^2{}_{21} &= \tfrac{1}{2}g^{12}g_{11,2} + \tfrac{1}{2}g^{22}g_{22,1} \\ \Gamma^1{}_{22} &= \tfrac{1}{2}g^{12}g_{22,2} + g^{11}g_{12,2} - \tfrac{1}{2}g^{11}g_{22,1} \\ \Gamma^2{}_{22} &= \tfrac{1}{2}g^{22}g_{22,2} + g^{12}g_{12,2} - \tfrac{1}{2}g^{12}g_{22,1} \end{aligned}\right\} \tag{17.1}$$

while in alphabetical notation[11]

$$\left.\begin{aligned} \Gamma^1{}_{11} &= g^{-1}\big\{ \quad \tfrac{1}{2}GE_u - FF_u + \tfrac{1}{2}FG_v\big\} \\ \Gamma^2{}_{11} &= g^{-1}\big\{ -\tfrac{1}{2}FE_u + EF_u - \tfrac{1}{2}EG_v\big\} \\ \Gamma^1{}_{12} = \Gamma^1{}_{21} &= g^{-1}\big\{ \quad \tfrac{1}{2}GE_v - \tfrac{1}{2}FG_u\big\} \\ \Gamma^2{}_{12} = \Gamma^2{}_{21} &= g^{-1}\big\{ -\tfrac{1}{2}FE_v + \tfrac{1}{2}EG_u\big\} \\ \Gamma^1{}_{22} &= g^{-1}\big\{ -\tfrac{1}{2}FG_v + GF_v - \tfrac{1}{2}GG_u\big\} \\ \Gamma^2{}_{22} &= g^{-1}\big\{ \quad \tfrac{1}{2}EG_v - FF_v + \tfrac{1}{2}FG_u\big\} \end{aligned}\right\} \tag{17.2}$$

In Riemannian geometry—which originated in the thesis (1854) of the 28-year-old Bernard Riemann (1826–1866), written at the request of Gauss—the curvature of n-dimensional manifolds is most commonly described by an object

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho{}_{\nu\sigma} - \partial_\nu\Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}$$

that is often called the "Riemann curvature tensor" but, since almost no equations appear in the text of Riemann's thesis,[12] is more properly called the

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[10] See again the Gauss equations on page 4.

[11] Use

$$\mathbb{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathbb{G}^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{pmatrix} = g^{-1}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

where now $g = \det\mathbb{G} = EG - F^2$.

[12] "On the hypotheses which lie at the bases of geometry." An English translation by W. K. Clifford is available on the web.

RIEMANN-CHRISTOFFEL TENSOR. So richly endowed with symmetry relations (including the 3-member symmetry $R_{\rho\sigma\mu\nu}+R_{\rho\mu\nu\sigma}+R_{\rho\nu\sigma\mu}=0$ called the "first Bianchi identity") is $R_{\rho\sigma\mu\nu}$ that in n dimensions only

$$\# = \tfrac{1}{12}n^2(n^2-1)$$

of its n^4 components are independent: as n ranges on $\{2,3,4,5,\ldots\}$ # ranges on $\{1,6,20,50,\ldots\}$. In the case $n=2$ of immediate interest, only one component is independent: all $R_{\rho\sigma\mu\nu}$ (indices ranging on $\{1,2\}$) are either 0 or $\pm R_{1212}$. Indeed, one in that case has

$$R_{\rho\sigma\mu\nu} = K(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$$

from which it follows in particular that

$$\begin{aligned} R_{1212} = K(g_{11}g_{22} - g_{12}g_{21}) &= K\cdot g \\ K &= \frac{R_{1212}}{g} \end{aligned} \tag{18.1}$$

which serves quite elegantly to describe K in terms of the metric and its first and second derivatives. One implication, by (15.1), is that

$$R_{1212} = ef - g^2$$

(here g has reverted to its former meaning) provides an intrinsic metric description of an expression derived from the extrinsic *second* fundamental form. The formula (18) is, however, quite inefficient from a computational point of view.[13] Much more efficient are this intrinsic formula due to Liouville[14]

$$K = \frac{1}{\sqrt{g}}\Big[\Big(\frac{\sqrt{g}}{E}\Gamma^2{}_{11}\Big)_v - \Big(\frac{\sqrt{g}}{E}\Gamma^2{}_{12}\Big)_u\Big] \tag{18.2}$$

and Brioschi's formula

$$K = \frac{D_1 - D_2}{g^2} \tag{18.3}$$

where

$$D_1 = \det\begin{pmatrix} -\frac{1}{2}E_{vv}+F_{uv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u-\frac{1}{2}E_v \\ F_v-\frac{1}{2}G_u & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

$$D_2 = \det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

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[13] Evaluation of $g_{1\rho}(\Gamma^\rho{}_{1\lambda}\Gamma^\lambda{}_{22}-\Gamma^\rho{}_{2\lambda}\Gamma^\lambda{}_{12})$ entails summing 23 5-term products, to which the evaluation of $g_{1\rho}(\partial_1\Gamma^\rho{}_{22}-\partial_2\Gamma^\rho{}_{12})$ contributes an additional 30 4-term products. Great simplifications can be expected to occur, however, in cases where significantly many of the Christoffel symbols either vanish or assume simple functional forms.

[14] Presented as equation (1.13) in Rogers & Schief. See also the "Gaussian curvature" article in Worlfram World.

Brioschi's formula is written out in detail in the Worlfram World article cited above. Somewhat simpler is the formula

$$K = -\frac{1}{4g^2}\begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} - \frac{1}{2\sqrt{g}}\Big\{\frac{\partial}{\partial v}\frac{E_v - F_u}{\sqrt{g}} - \frac{\partial}{\partial u}\frac{F_v - G_u}{\sqrt{g}}\Big\} \tag{18.4}$$

that appears on page 20 in Chapter 4 of CFT (undated Classical Field Theory notes). On coordinate patches where the coordinates are orthogonal ($F = 0$) (18.4) reduces to

$$K = -\frac{1}{2\sqrt{EG}}\Big(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\Big) \tag{18.5}$$

Finally, for surfaces Σ presented extrinsically as the graph of a function

$$z = Z(x, y)$$

one has (compare (9.1))

$$K = \frac{Z_{xx}Z_{yy} - Z_{xy}^2}{(1 + Z_x^2 + Z_y^2)^2} \tag{18.6}$$

PART TWO

Differential Geometry of some Specific Surfaces

Sphere. We look first to the sphere because it provides the simplest possible laboratory in which to illustrate the meanings of key concepts and to demonstrate the accuracy and effectiveness of various formulae. The sphere of radius ρ is, relative to a Cartesian frame in $\mathbb{R}^3$, defined implicitly by

$$x^2 + y^2 + z^2 - \rho^2 = 0$$

explicitly (upper and lower hemispheres separately) by

$$z = \pm\sqrt{\rho^2 - x^2 - y^2}$$

and parametrically (in the most common parameterization) by

$$\boldsymbol{r}(u, v) = \begin{pmatrix} \rho\,\cos u\,\cos v \\ \rho\,\cos u\,\sin v \\ \rho\,\sin u \end{pmatrix}$$

3D figures are constructed by the *Mathematica* commands

`ContourPlot3D[`*etc.*`]`
`Plot3D[`*etc.*`, AspectRatio → Automatic]`
`ParametricPlot3D[`*etc.*`]`

respectively. From

$$\boldsymbol{r}_u = \begin{pmatrix} -\rho\,\sin u\,\cos v \\ -\rho\,\sin u\,\sin v \\ \rho\,\cos u \end{pmatrix}, \qquad \boldsymbol{r}_v = \begin{pmatrix} -\rho\,\cos u\,\sin v \\ \rho\,\cos u\,\cos v \\ \rho\,\sin u \end{pmatrix}$$

we obtain the coefficients of the 1$^{\text{st}}$ fundamental form (components of the metric matrix)

$$\begin{aligned}\boldsymbol{r}_u\cdot\boldsymbol{r}_u &= E = \rho^2\\ \boldsymbol{r}_u\cdot\boldsymbol{r}_v &= F = 0\\ \boldsymbol{r}_v\cdot\boldsymbol{r}_v &= G = \rho^2\cos^2 u\end{aligned}$$

Major simplifications result from the circumstances that E is constant and $F = 0$ (the coordinates are orthogonal, except at the poles). From Liouville's (18.5) it follows immediately that

$$K = 1/\rho^2 \quad : \quad \text{everywhere (all } u, v)$$

Looking to (17.2) we find the Christoffel symbols to be given in this instance by

$$\begin{aligned}\Gamma^1{}_{11} &= g^{-1}\big\{\quad \tfrac{1}{2}GE_u\big\} = 0\\ \Gamma^2{}_{11} &= g^{-1}\big\{-\tfrac{1}{2}EG_v\big\} = 0\\ \Gamma^1{}_{12} = \Gamma^1{}_{21} &= g^{-1}\big\{\quad \tfrac{1}{2}GE_v\big\} = 0\\ \Gamma^2{}_{12} = \Gamma^2{}_{21} &= g^{-1}\big\{\quad \tfrac{1}{2}EG_u\big\} = -\tan u\\ \Gamma^1{}_{22} &= g^{-1}\big\{-\tfrac{1}{2}GG_u\big\} = \cos u\, \sin u\\ \Gamma^2{}_{22} &= g^{-1}\big\{\quad \tfrac{1}{2}EG_v\big\} = 0\end{aligned}$$

It is for the sphere geometrically obvious that $\boldsymbol{N}$ is parallel to $\boldsymbol{r}$, but the cross product requires that we introduce a minus sign:

$$\boldsymbol{N}(u,v) = -\frac{\boldsymbol{r}(u,v)}{|\boldsymbol{r}(u,v)|} = -\begin{pmatrix}\cos u\, \cos v\\ \cos u\, \sin v\\ \sin u\end{pmatrix}$$

The coefficients of the 2^{st} fundamental form are found therefore to be given by

$$\begin{aligned}\boldsymbol{r}_{uu}\cdot\boldsymbol{N} &= e = \rho\\ \boldsymbol{r}_{uv}\cdot\boldsymbol{N} &= f = 0\\ \boldsymbol{r}_{vv}\cdot\boldsymbol{N} &= g = \rho\cos^2 u\end{aligned}$$

which information places us (by (15)) in position to write

$$K = \frac{eg - f^2}{EG - F^2} = \frac{1}{\rho^2}$$

$$K_m = \frac{Eg - 2Ff + Ge}{EG - F^2} = \frac{1}{\rho}$$

which by (9) give

$$\kappa_1 = \kappa_2 = K_m \pm \sqrt{0} = \frac{1}{\rho}$$

Recall that for plane curves one has

$$\kappa = \frac{1}{\text{radius of curvature } \rho} = \frac{1}{\text{radius of osculating circle}}$$

For surfaces it make sense to speak of a "radius of the osculating sphere" *only* when $\kappa_1 = \kappa_2$.

The Gauss equations (4.2) have become

$$\begin{aligned}\boldsymbol{r}_{uu} &= \rho\boldsymbol{N}\\ \boldsymbol{r}_{uv} &= -\tan u\cdot\boldsymbol{r}_v\\ \boldsymbol{r}_{vv} &= \cos u\,\sin u\cdot\boldsymbol{r}_u + \rho\cos^2 u\,\boldsymbol{N}\end{aligned}$$

which are found by computation to be correct, while the Weingarten equations (5) have become

$$\begin{aligned}\boldsymbol{N}_u &= \frac{e}{E}\boldsymbol{r}_u = -\frac{1}{\rho}\,\boldsymbol{r}_u\\ \boldsymbol{N}_v &= \frac{g}{G}\boldsymbol{r}_v = -\frac{1}{\rho}\,\boldsymbol{r}_v\end{aligned}$$

which are obvious corollaries of $\boldsymbol{N} = -\boldsymbol{r}/\rho$. Tedious computation[15] confirms finally that

$$R_{1212} = \rho^2\cos^2 u$$

which by (18.1) gives back $K = 1/\rho^2$.

Surfaces of revolution, ruled surfaces & hyperboloids of a single sheet. We look next to surfaces defined implicitly by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

the z-sections of which are elliptical

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$$

and when $a = b$ become circular

$$x^2 + y^2 = a^2\Big(1 + \frac{z^2}{c^2}\Big) \tag{19}$$

Hyperboloids of the latter type are SURFACES OF REVOLUTION (about the z-axis), and admit of the natural parameterization

$$\boldsymbol{r}(u,v) = \begin{pmatrix} f(u)\cos v\\ f(u)\sin v\\ u\end{pmatrix} \quad\text{with}\quad f(u) = a\Big(1 + \frac{u^2}{c^2}\Big)^{\frac{1}{2}} \tag{20.1}$$

Alternatively (and just as naturally), we might write

$$\boldsymbol{r}(u,v) = \begin{pmatrix} u\\ v\\ (c/a)\sqrt{u^2+v^2-a^2}\end{pmatrix} \tag{20.2}$$

[15] See CFT pages 37–38. The MTW cited there is a reference to Misner, Thorne & Wheeler, *Gravitation* (1973).

Working first from (20.1), we find

$$\begin{aligned}
E &= \boldsymbol{r}_u\boldsymbol{\cdot}\boldsymbol{r}_u = \frac{a^2u^2+c^2u^2+c^4}{c^2(u^2+c^2)}\\
F &= \boldsymbol{r}_u\boldsymbol{\cdot}\boldsymbol{r}_v = 0\\
G &= \boldsymbol{r}_v\boldsymbol{\cdot}\boldsymbol{r}_v = \frac{a^2(u^2+c^2)}{c^2}
\end{aligned}$$

Since $F=0$ we can use (18.5) to compute

$$K = -\frac{c^6}{(a^2u^2+c^2u^2+c^4)^2} \tag{21}$$

The unit normal is given by

$$\boldsymbol{N} = \frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{|\boldsymbol{r}_u\times\boldsymbol{r}_v|} = \frac{c^2}{a\sqrt{a^2u^2+c^2u^2+c^4)^2}}\begin{pmatrix} -a\sqrt{1+(u/c)^2}\cos v\\ -a\sqrt{1+(u/c)^2}\sin v\\ a^2u/c^2\end{pmatrix}$$

so we have

$$\begin{aligned}
e &= \boldsymbol{r}_{uu}\boldsymbol{\cdot}\boldsymbol{N} = -\frac{ac^2}{(u^2+c^2)\sqrt{a^2u^2+c^2u^2+c^4}}\\
f &= \boldsymbol{r}_{uv}\boldsymbol{\cdot}\boldsymbol{N} = 0\\
g &= \boldsymbol{r}_{vv}\boldsymbol{\cdot}\boldsymbol{N} = \frac{a(u^2+c^2)}{\sqrt{a^2u^2+c^2u^2+c^4}}
\end{aligned}$$

Gauss' construction (15.1) now gives

$$K = \frac{eg-f^2}{EG-F^2} = -\frac{c^6}{(a^2u^2+c^2u^2+c^4)^2}$$

in precise agreement with (21).

From the fact that K is everywhere negative[16] we infer that the principal curvatures κ_1 and κ_2 are everywhere of opposite signs: every point on the hyperboloid is a saddlepoint. This traces to the circumstance that the eigenvalues of

$$\mathbb{H}(u,v) = \begin{pmatrix} e & 0\\ 0 & g\end{pmatrix}$$

are of opposite signs.

Because $F=f=0$ and the remaining coefficients $\{E,G,e,g\}$ are all v-independent, the hyperboloidal Christoffel symbols (17.2) are subject in this coordinate system to simplifications quite like those we encountered formerly in the spherical case.

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[16] At the "throat" of the hyperboloid (*i.e.*, at $u=0$) the Gaussian curvature $K=-1/c^2$. Asymptotically it approaches 0 as $[-a^6/(a^2+c^2)^2]\cdot u^{-4}$.

The seemingly simpler parameterization (20.2) is found to lead to expressions that are too complicated to be informative.[17]

> MORAL: An ill-chosen parameterization can greatly complicate things, and an orthogonal parameterization (when possible) can be expected to purchase major simplifications.

A change of variables $u \to c\sinh w$ brings (20.1) to the rather more attractive form

$$\boldsymbol{r} = \begin{pmatrix} a\cosh w\cos v \\ a\cosh w\sin v \\ c\sinh w \end{pmatrix}$$

RULED SURFACES are produced by waving straight lines (in this context called "rules") around in 3-space. This can be accomplished analytically by letting $\boldsymbol{c}(u)$—called the "directix"—trace a space curve $\mathcal{C}$ in 3-space, attaching a "director" vector $\boldsymbol{d}(u)$ to each of the points of $\mathcal{C}$ and forming

$$\boldsymbol{r}(u,v) = \boldsymbol{c}(u) + v\,\boldsymbol{d}(u)$$

I illustrate the idea as it pertains to the hyperboloid of a single sheet, perhaps the most familiar of all ruled surfaces. Let $\mathcal{C}$ be a circle of radius a, inscribed on the xy-plane with center at the origin:

$$\boldsymbol{c}(u) = \begin{pmatrix} a\cos u \\ a\sin u \\ 0 \end{pmatrix}$$

Let $\boldsymbol{d}(u)$ stand normal to $\boldsymbol{c}(u)$:

$$\boldsymbol{d}(u) = \begin{pmatrix} \sin u \\ -\cos u \\ k \end{pmatrix}$$

Then

$$\boldsymbol{r}(u,v) = \begin{pmatrix} a\cos u + v\sin u \\ a\sin u - v\cos u \\ vk \end{pmatrix}$$

and we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 + \frac{v^2}{a^2} \quad\text{and}\quad 1 + \frac{z^2}{c^2} = 1 + \frac{k^2v^2}{c^2}$$

We recover (19) when we set

$$k = \pm c/a$$

The hyperboloid of a single sheet is therefore (famously) *doubly* ruled:

$$\boldsymbol{r}_{\pm}(u,v) = \begin{pmatrix} a\cos u + v\sin u \\ a\sin u - v\cos u \\ \pm cv/a \end{pmatrix} \tag{22.1}$$

Working from the ruled parameterization (22.1) we obtain

[17] The parameterization (20.2) does, however, become tractable when one sets $a = c = 1$.

$$
\begin{aligned}
E &= v^2 + a^2\\
F &= -a\\
G &= (a^2 + c^2)/a^2
\end{aligned}
$$

$$
\boldsymbol{N} = \frac{c}{\sqrt{(a^2+c^2)v^2 + a^2c^2}}\begin{pmatrix} a\cos u + v\sin u\\ a\sin u - v\cos u\\ -(a/c)v\end{pmatrix}
$$

$$
\begin{aligned}
e &= -\frac{c(v^2+a^2)}{\sqrt{(a^2+c^2)v^2+a^2c^2}}\\
f &= \frac{ac}{\sqrt{(a^2+c^2)v^2+a^2c^2}}\\
g &= 0
\end{aligned}
$$

The latter equation joins the circumstance that $\{E, F, G, e, f, g\}$ are all u-independent to simplify construction (17.2) of the Christoffel symbols, but because $F \neq 0$ (the coordinates are not orthogonal) Liouville's (18.2) is not applicable. But Gauss' construction (15.1) supplies

$$
K = -\frac{a^4c^2}{((a^2+c^2)v^2+a^2c^2)^2}
$$

Reverting from the ruled parameterization (22.1_+) in which $z = cv/a$ to the parameterization (20.1) in which $z = u$, we in the preceding equation make the replacement $v \to au/c$ and recover precisely the former description (21) of K.

Ruled hyperboloids with *elliptical* throats are obtain by setting

$$
\boldsymbol{r}_\pm(u,v) = \begin{pmatrix} a\cos u + va\sin u\\ b\sin u - vb\cos u\\ \pm vc\end{pmatrix}
$$

which gives back (22.1) in the case $b = a$ if in place of va one writes simply v.

Returning now to the parameterization (20.1)—which has been seen to give

$$
K = -\frac{c^6}{(a^2u^2 + c^2u^2 + c^4)^2}
$$

—the mean curvature becomes

$$
K_m = \frac{Eg - 2Ff + Ge}{EG - F^2} = \frac{c^2\big((a^2+c^2)u^2 - (a^2-c^2)c^2\big)}{a(a^2u^2 + c^2u^2 + c^4)^{\frac{3}{2}}}
$$

but—which is the point of this concluding remark—the principal curvatures

$$
\kappa = K_m \pm \sqrt{K_m^2 - K}
$$

are almost too complicated to write down, even though the hyperboloid is in other respects such a simple surface.

Conjugate directions, asymptotic directions and curves. The 1^{st} fundamental form gives rise to a positive-definite symmetric matrix $\mathbb{G}$ that describes the local metric structure of the surface Σ. The equation $\mathbb{G}\boldsymbol{x} = \lambda\boldsymbol{x}$ serves to associate a pair of orthogonal tangent vectors with every point P of Σ. The 2^{nd} fundamental form, which refers to the normal curvature (variation of $\boldsymbol{N}$) at P, gives rise on the other hand—as was remarked already on page 2—to a symmetric matrix $\mathbb{H}$ for which $\det\mathbb{H}$ can assume either sign, depending upon the local value of the Gaussian curvature K. For surfaces with everywhere-negative curvature it is everywhere the case that $\det\mathbb{H} < 0$ (eigenvalues of opposite sign).

Given a tangent vector $\boldsymbol{x}$ at P, the equation $(\boldsymbol{y}, \mathbb{H}\boldsymbol{x}) = 0$ serves to associate with $\boldsymbol{x}$ a vector (orthogonal to $\boldsymbol{x}$) that will be said to be "conjugate" to $\boldsymbol{x}$. Suppose, for example, we have arranged to have

$$\mathbb{H} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and agreed (by suspending any interest in normalization) to write

$$\boldsymbol{x} = \begin{pmatrix} x \\ 1 \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y \\ 1 \end{pmatrix}$$

Then

$$(\boldsymbol{y}, \mathbb{H}\boldsymbol{x}) = \lambda_1 xy + \lambda_2 = 0 \quad \Longrightarrow \quad y = -\frac{\lambda_2}{x\,\lambda_1}$$

The tangent vector $\boldsymbol{x}$ is said to be "self conjugate" (or "asymptotic") if $\boldsymbol{y} = \boldsymbol{x}$, which entails

$$x = \pm\sqrt{-\lambda_2/\lambda_1} \quad : \quad \text{real if and only if } K < 0$$

The implication is that at points P on surfaces Σ of negative curvature there are typically *two* asymptotic directions.

Curves $\mathcal{C}$ inscribed on Σ that are endowed with the property that all the tangents are asymptotic are called are called ASYMPTOTIC CURVES. To illustrate this notion I return again to the hyperboloid (20.1), where

$$\mathbb{H} = \begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix} \quad \text{with} \quad \begin{cases} e = -\dfrac{ac^2}{(u^2+c^2)\sqrt{a^2u^2+c^2u^2+c^4}} \\[2ex] g = \dfrac{a(u^2+c^2)}{\sqrt{a^2u^2+c^2u^2+c^4}} \end{cases}$$

The differential tangent vector $\begin{pmatrix} du \\ dv \end{pmatrix}$ will be asymptotic if and only if

$$\frac{du}{dv} = \pm\sqrt{-g/e} = \pm\frac{u^2+c^2}{c}$$

which entails

$$u(v) = \pm c\tan(v - v_0) \quad \Longleftrightarrow \quad v(u) = \pm\arctan(u/c) + v_0 \tag{23}$$

where v_0 is a constant of integration. Returning with this information to (20.1) and noting that

$$f(u) = a\Big(1 + \frac{u^2}{c^2}\Big)^{\frac{1}{2}} = a\sec[\arctan(u/c)]$$

we have

$$\boldsymbol{r} = \begin{pmatrix} a\sec[\arctan(u/c)]\cos v \\ a\sec[\arctan(u/c)]\sin v \\ u \end{pmatrix}$$

which by (23) and a pair of trigonometric identities becomes

$$\boldsymbol{r}_\mp = \begin{pmatrix} a\cos v_0 \mp u\cdot(a/c)\sin v_0 \\ a\sin v_0 \pm u\cdot(a/c)\cos v_0 \\ u \end{pmatrix}$$

This—remarkably, and quite unexpectedly—is seen to present the hyperboloid as a ruled surface: to recover (22.1) replace $v_0 \to u$, $u \to \mp cv/a$. The asymptotic curves are in this instance rectilinear *rules*. Less trivial examples of aymptotic curves will be forthcoming.

The hexenhut. In looking to the seldom encountered surface defined implicitly by the cubic equation

$$x^3 + y^3 + z^3 - 3xyz = 1 \tag{24}$$

I am allowing myself a sentimental indulgence, for it was (see again page 1) this oddly but aptly named surface that led me to undertake this excursion into the theory of surfaces.

Graphic experimentation shows the hexenhut (24) to be a surface of revolution with axis coincident with the principal diagonal $\{0,0,0\} \to \{1,1,1\}$ of the unit cube. The surface is easier to comprehend (and to work with) if one rotates[18]

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbb{R}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad : \quad \mathbb{R} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

to a different orthogonal frame in 3-space. In $\{X, Y, Z\}$ variables (24) reads

$$Z(X^2 + Y^2) = \frac{2}{3\sqrt{3}} \equiv \alpha^2 = 0.3849$$

which clearly describes a surface of revolution about the Z-axis. We have the immediate parameterization

$$\boldsymbol{r} = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ u \end{pmatrix} \quad \text{with} \quad f(u) = \alpha/\sqrt{u} \tag{25}$$

---

[18] I am indebted here to Ahmed Sebbar. *Mathematica* confirms that $\mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{I}$.

From (25) we obtain

$$\begin{aligned}E &= 1 + \frac{\alpha^2}{4u^3}\\ F &= 0\\ G &= \frac{\alpha^2}{u}\\ e &= -\frac{3\alpha^2}{2u\sqrt{4u^3\alpha^2+\alpha^4}}\\ f &= 0\\ g &= \frac{2u\alpha^2}{\sqrt{4u^3\alpha^2+\alpha^4}}\end{aligned}$$

The Gaussian curvature is given therefore (since $F = f = 0$) by

$$K = \frac{eg}{EG} = -\frac{12u^4}{(4u^2+\alpha^2)^2} \tag{26}$$

while (18.5) reduces (since all coefficients are v-independent) to

$$K = -\frac{1}{2\sqrt{EG}}\Big(\frac{G_u}{\sqrt{EG}}\Big)_u = \text{same thing}$$

The curvature $K(u)$ is 0 at $u = 0$ (*i.e.*, at the infinitely remote almost-flat brim of the hexenhut), drops to -1 at $u = (\frac{1}{2}\alpha^2)^{\frac{1}{3}} = 1/\sqrt{3}$ and approaches 0 as $u \to \infty$ (*i.e.*, as one approaches the ever-more-nearly-cylindrical tip of the hexenhut).

For asymptotic curves we have again (because $f = 0$) the differential equation $du/dv = \pm\sqrt{-g/e}$, or more conveniently

$$\frac{dv}{du} = \pm\sqrt{-e/g} = \pm\beta u^{-1} \quad : \quad \beta = \tfrac{\sqrt{3}}{2}$$

which gives

$$v(u) = \pm\beta\log(u/u_0)$$

where u_0 is a constant of integration. Asymptotic curves inscribed on the hexenhut acquire thus the description

$$\boldsymbol{r}_\pm(u) = \begin{pmatrix} f(u)\cos[\beta\log(u/u_0)]\\ \pm f(u)\sin[\beta\log(u/u_0)]\\ u\end{pmatrix} \quad \text{with} \quad f(u) = \alpha/\sqrt{u} \tag{27}$$

where it is the value of u_0 that distinguishes one such curve from another; *i.e.*, that assigns a "name" to each individual asymptotic curve of specified type. The hexenhut emerges as the envelope of the u_0-parameterized family of such curves (either type, or both).

The Christoffel symbols associated with the $\{u, v\}$-parameterization (25) of the hexenhut simplify for precisely the reasons stated already at the bottom of page 17.

The pseudosphere. "Coordinate geometry" or "analytic geometry," the fusion of geometry and calculus made possible by Descartes' introduction of a coordinate system onto the Euclidean plane, gave rise in the 16$^{\text{th}}$ Century to a flurry of activity relating to the properties of plane curves and of curves (evolute, involute, pedal, *etc.*) derived from curves.[19]

The TRACTRIX, or "curve of pursuit," was introduced by Claude Perrault (1670), and its properties studied by Newton (1676) and Huygens (1692). In Lockwood[19] it is discussed (Chapter 13, pages 118–124) in conjunction with the CATENARY,[20] its evolute (envelope of its normals: here enters the theory of Legendre transformations), of which it is therefore the involute.

The "curve of pursuit" acquires its name from the following consideration: A string of length ρ is looped around the z-axis at $\{0,0\}$ and attached at its other end to a sled at $\{\rho,0\}$. As the loop ascends the z-axis the dragged sled traces a curve $z(x)$ with the property that the length of the tangent at $\{x,z(x)\}$ to the z-intercept is ρ. Working from a sketch, one has

$$\frac{dz}{dx} = -\frac{\sqrt{\rho^2-x^2}}{x} \quad \text{with} \quad z(\rho)=0$$

of which the solution is

$$\begin{aligned} z(x) &= \rho\log\frac{\rho+\sqrt{\rho^2-x^2}}{x} - \sqrt{\rho^2-x^2} \\ &= \rho\,\text{arcsech}(x/\rho) - \sqrt{\rho^2-x^2} \end{aligned}$$

By a change of variables

$$x = \rho\,\text{sech}\,u$$

we obtain

$$z = \rho u - \rho\tanh u$$

Revolution about the z-axis generates a surface

$$\boldsymbol{r}(u,v) = \begin{pmatrix} \rho\,\text{sech}\,u\,\cos v \\ \rho\,\text{sech}\,u\,\sin v \\ \rho u - \rho\tanh u \end{pmatrix} \tag{28}$$

that when plotted looks to be a "hexenhut with a vengence." It might plausibly be called a "tractrixoid," but was in fact called a "tractricold" until given the name PSEUDOSPHERE by Beltrami in 1868. Elementary properties of the

[19] See E. H. Lockwood's wonderful *A Book of Curves* (1961), which is now available on the web as a free pdf download.

[20] Architectural applications of the catenary were discussed by Robert Hooke in the 1670s, and its mathematical properties had been worked out by Leibniz, Huygens and Johann Bernouli by 1691. It arose as a refinement of Galileo's observation that the profile of a hanging chain is approximately parabolic, and was given its name—again in an architectural connection—by Thomas Jefferson, in a letter to Thomas Payne.

pseudosphere had been worked out by Huygens in 1693, who found the surface area to be that of a sphere of radius ρ, and the enclosed volume to be half that of such a sphere. But recognition of the most interesting properties of the pseudospherical surface Σ had to await the development of differential geometry in the 19^{th} Century, and it is to some of those that we now turn.

Working from (28) we find

$$\left.\begin{aligned} E &= \rho^2 \tanh^2 u \\ F &= 0 \\ G &= \rho^2 \operatorname{sech}^2 u \end{aligned}\right\} \tag{29.1}$$

$$\boldsymbol{N} = -\begin{pmatrix} \tanh u\, \cos v \\ \tanh u\, \sin v \\ \operatorname{sech} u \end{pmatrix}$$

$$\left.\begin{aligned} e &= -\rho \operatorname{sech} u \tanh u \\ f &= 0 \\ g &= +\rho \operatorname{sech} u \tanh u \end{aligned}\right\} \tag{29.2}$$

Whether we return with the three relations (29.1) to Liouville's (18.5) or with all six relations (29) to Gauss' (15.5), we obtain in either event

$$K = -\frac{1}{2\sqrt{EG}}\Big(\frac{G_u}{\sqrt{EG}}\Big)_u = \frac{eg}{EG} = -\frac{1}{\rho^2} \tag{30}$$

which exposes the most salient property of the pseudosphere (and accounts for its name): it *has at all points the <u>same</u> negative curvature*.

We note in passing that in this parameterization of the pseudosphere the Christoffel symbols (17.2) simplify for precisely the reason remarked already on pages 17 and 22.

Looking now to the pseudospheric asymptotic curves, we have

$$\frac{dv}{du} = \pm\sqrt{-g/e} = \pm 1 \quad\Longrightarrow\quad v = u_0 \pm u$$

where u_0 is again a constant of integration. The following equations serve therefore to inscribe on the pseudosphere two populations of asymptotic curves:

$$\boldsymbol{r}_+(u) = \begin{pmatrix} \rho \operatorname{sech} u\, \cos(u_0 + u) \\ \rho \operatorname{sech} u\, \sin(u_0 + u) \\ \rho u - \rho \tanh u \end{pmatrix}, \quad \boldsymbol{r}_-(u) = \begin{pmatrix} \rho \operatorname{sech} u\, \cos(u_0 - u) \\ \rho \operatorname{sech} u\, \sin(u_0 - u) \\ \rho u - \rho \tanh u \end{pmatrix}$$

A unified "asymptotic parameterization of the pseudosphere" is achieved if one introduces new parameters $\{x, y\}$—not to be confused with Cartesian coordinates—by

$$\begin{aligned} u &= x + y \\ v &= x - y \end{aligned}$$

Then (28) becomes

$$\boldsymbol{r}(x,y) = \begin{pmatrix} \rho\,\text{sech}(x+y)\,\cos(x-y) \\ \rho\,\text{sech}(x+y)\,\sin(x-y) \\ \rho\,(x+y) - \rho\tanh(x+y) \end{pmatrix} \tag{31}$$

which for y fixed and x variable produces a y-paramterized family of $\boldsymbol{r}_+$ curves, and for x fixed and y variable produces an x-paramterized family of $\boldsymbol{r}_-$ curves.

We do now a non-obvious thing: working from (31), we look to

$$\omega(x,y) = \begin{cases} \text{angle subtended by the asymptotic} \\ \text{curves that intersect at the } \{x,y\} \end{cases}$$

Noting that the tangent vectors

$$\begin{aligned} \boldsymbol{R}_x &= \rho^{-1}\partial_x \boldsymbol{r} \\ \boldsymbol{R}_y &= \rho^{-1}\partial_y \boldsymbol{r} \end{aligned}$$

are unit vectors, we have

$$\begin{aligned} \omega(x,y) = \arccos(\boldsymbol{R}_x\cdot \boldsymbol{R}_y) &= \arccos\big[1 - 2\,\text{sech}^2(x+y)\big] \\ &= \arccos\Big[1 - \frac{4}{1+\cosh(2x+2y)}\Big] \end{aligned}$$

We are informed by *Mathematica*[21] that, as was first noticed by Edmond Bour[22] (1862), we have on one hand

$$\omega_{xy}(x,y) = 2\,\text{sech}(x+y)\tanh(x+y)$$

and on the other hand

$$\sin\omega = 2\,\text{sech}(x+y)\tanh(x+y) \tag{32}$$

Thus did the SINE-GORDON EQUATION[23]

$$\partial_x\partial_y\,\omega = \sin\omega \tag{33}$$

enter the literature of mathematics, fully half a century before it became central to the physical theory of solitons.

I present now a sneeky, indirect alternative derivation of (33). It proceeds from Liouville's formula (18.2), which was recommended on the ground that

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[21] *Mathematica* has contributed indispensably to all the work reported in the present essay. One can only admire the patience (and accuracy) of the pioneers who were obliged to do all the heavy calculation by hand.

[22] The diffential geometry of the pseudosphere was pioneered by Ferdinand Minding (1806–1885), whose ideas were taken up and cultivated by Bour (1832–1866), whose work stimulated that of (among others) Eugenio Beltrami (1835–1900) and Albert Bäcklund (1845–1922).

[23] See the companion essay, "Some remarks concerning the sine-Gordon equation" (November, 2015).

it draws only upon the local metric structure of a surface (information written into the 1st fundamental form), and which I repeat:

$$K = \frac{1}{\sqrt{g}}\Big[\Big(\frac{\sqrt{g}}{E}\Gamma^2{}_{11}\Big)_y - \Big(\frac{\sqrt{g}}{E}\Gamma^2{}_{12}\Big)_x\Big]$$

Working from (31)—which insofar as it alludes to the asymptotic direction concept is a child of the 2nd fundamental form—we find

$$\begin{aligned} E &= \rho^2 \\ F &= \rho^2\big[1 - 2\,\mathrm{sech}^2(x+y)\big] \\ G &= \rho^2 \end{aligned}$$

which by (17.2) entail

$$\begin{aligned} \Gamma^2{}_{11} &= g^{-1}EF_x \\ \Gamma^2{}_{12} &= 0 \end{aligned}$$

since $E_x = E_y = G_x = G_y = 0$. So Liouville's formula has assumed the simple form

$$K = \frac{1}{\sqrt{g}}\Big(\frac{\sqrt{g}}{E}\Gamma^2{}_{11}\Big)_y \tag{34}$$

But $g = EG - F^2 = 4\rho^4\,\mathrm{sech}^2(x+y)\tanh^2(x+y)$ so

$$\left.\begin{aligned} \sqrt{g} &= 2\,\rho^2\,\mathrm{sech}(x+y)\tanh(x+y) \\ \Gamma^2{}_{11} &= \coth(x+y) \end{aligned}\right\} \implies \frac{\sqrt{g}}{E}\Gamma^2{}_{11} = 2\,\mathrm{sech}(x+y)$$

and (34) reads

$$K = \frac{1}{2\,\rho^2\,\mathrm{sech}(x+y)\tanh(x+y)}\,\partial_y\Big(2\,\mathrm{sech}(x+y)\Big) = -\frac{1}{\rho^2}$$

It is gratifying (if certainly no surprise) that Liouville's formula has returned this familiar result. What *is* surprising is the news that its success hinges in this instance on the elementary statement

$$\partial_y\Big(2\,\mathrm{sech}(x+y)\Big) = -2\,\mathrm{sech}(x+y)\tanh(x+y) \tag{35}$$

The sine-Gordon equation now follows from the non-obvious observations that

$$\begin{aligned} 2\,\mathrm{sech}(x+y) &= \partial_x\Big[4\arctan\Big(\tanh\frac{x+y}{2}\Big)\Big] \\ \sin\Big[4\arctan\Big(\tanh\frac{x+y}{2}\Big)\Big] &= 4\,e^{x+y}\,\frac{e^{2(x+y)}-1}{(e^{2(x+y)}+1)^2} = 2\,\mathrm{sech}(x+y)\tanh(x+y) \end{aligned}$$

which bring (35) to the form

$$\partial_x\partial_y\Omega = -\sin\Omega \quad\text{where}\quad \Omega(x,y) = 4\arctan\Big(\tanh\frac{x+y}{2}\Big) \tag{36}$$

We recover (33) by observing (graphically) that

$$\arccos\big[1-2\,\text{sech}^2x\big]=\begin{cases}4\arctan\big(\tanh\frac{1}{2}x\big)+\pi & : \quad x<0\\ \pi-4\arctan\big(\tanh\frac{1}{2}x\big) & : \quad x>0\end{cases}$$

Equation (33) is unchanged when the sign of Ω is reversed; the additive π terms are invisible to the $\partial\partial$ on the left, but reverse the sign of the sine on the right.

Since x and y enter into $\Omega(x,y)$—as also into $\omega(x,y)$—only through their sum, we have $\partial_x\partial_x = \partial_x\partial_y = \partial_y\partial_y$; it was by arbitrary selection that we elected to write $\partial_x\partial_y$ in (33) and (36).[24] This situation is, however, atypical of sine-Gordon theory, and does not pertain to (for example) the function

$$\omega(x,y)=4\arctan\big(e^{ax+\frac{1}{a}y}\big)$$

encountered at (8.1) in the companion essay cited on page 25, for which one has

$$\omega_{xy}=\sin\omega \quad \text{but} \quad \omega_{xx}-a^4\omega_{yy}=0$$

Concluding comments. Since it has best served my purpose—which has been to prepare myself for deeper penetration into the material presented in the several-times-mentioned monograph by Rogers & Schief—and anyway conforms most comfortably to my admittedly old-fashioned way of thinking about mathematics, I offer no apology for the fact that I have adhered here to classical methods and notations, and have provided no hint of the high level of abstraction with which the topics treated here tend to be treated in the modern literature, and even in Wikipedia articles.[25] I do, however, apologize to my reader (should ever I enjoy the company of one) for the oppressive degree of detail with which I have treated some topics; the simple fact is that I have written not for the edification of a reader, but to provide myself with a record of my thought as I worked through issues that initially confused me.

In the course of the work I have become aware of some of the many often excellent differential geometric course notes that mathematicians scattered about the world have generously placed on the web. And I have been led to become the proud owner of a copy of the 1st edition of Luther Eisenhart's *Treatise on the Differential Geometry of Curves and Surfaces* (1909).

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[24] It would appear, therefore, that Bour gave birth in 1862 to triplets, of which only one can claim ancestry to the sine-Gordon equation; one would like to have access to the paper in which (if ever he did) he recognized the special importance of that particular child.

[25] An intermediate way station on the road to abstraction is marked by an account of "curvature" that was communicated to me by Thomas Wieting.

Figure Captions

Figure 1. Hyperboloid of a Single Sheet. The figure derives from the parametric representation (20.1), and portrays (as to all these hyperboloidal figures) a "unit hyperboloid" $(a = b = c = 1)$.

Figure 2. Ruled hyperboloid. The figure derives from the $\boldsymbol{r}_+$ of(22.1), with $u = \frac{2\pi}{37}n : n = 1, 2, \ldots, 38$, which produces the $\circlearrowright$ family of rules. The hyperboloid is bounded by coaxial hoops of radius R and vertical separation D. Counterrotation of the hoops (upper hoop in the $\circlearrowright$ direction) brings the surface to conical form; subsequent rotation in the $\circlearrowleft$ direction brings the surface through all possible (centrally circular) figures until it becomes cylindrical, then reverses sense and passes again through all figures until it is again conical. The minimal surface (soapfilm) supported by such a pair of hoops is (when D/R falls within a certain interval) known to be a catenoid of revolution, to which one of the hyperboloids provides a best approximation. One is reminded of Galileo's observation that the curve of a hanging chain is "approximately parabolic," an observation that soon led others to the discovery that it is in fact a catenary. One wonders why Galileo did not say "approximately hyperbolic." It is established in the text that hyperboloidal "rules" are "asymptotic curves."

Figure 3. Hexenhut in Natural Position. The figure derives from the implicit equation $x^3 + y^3 + z^3 - 3xyz = 1$.

Figure 4. Reoriented Hexenhut. The figure, derived from (25), displays the hexenhut as a surface of revolution about a reoriented Z-axis.

Figure 5. Hexenhut with Inscribed Asymptotic Curves. The figure, derived from this variant of the $\boldsymbol{r}_+$ of (27)

$$\boldsymbol{r}_+(u) = \begin{pmatrix} f(u)\cos[\beta\log(u) + \frac{2\pi}{25}n] \\ f(u)\sin[\beta\log(u) + \frac{2\pi}{25}n] \\ u \end{pmatrix} \quad \text{with} \quad \begin{cases} f(u) = \alpha/\sqrt{u} \\ n = 1, 2, \ldots, 26 \end{cases}$$

displays the hexenhut inscribed with one of its two families of asymptotic curves.

Figure 6. Complete Pseudosphere. The figure, derived from the parametric representation (28), displays both halves of of the pseudosphere, which—unlike the hexenhut—has a finite diameter, and encloses a finite volume.

Figure 7. Upper Hemisphere of a Pseudosphere. References to the pseudosphere are usually references to only half of the complete surface, a "hemipseudosphere"?

Figure 8. Pseudosphere with Inscribed Asymptotic Curves. The figure, derived from (31), presents the surface as the envelope of asymptotic curves, of which *Mathematica* has elected to display only some. Note that the angle subtended by intersecting asymptotes—to which Bour directed our attention—decreases as monotonically one ascends, from π at the base to 0 as $z \to \infty$.

Figure 1. Hyperboloid of a Single Sheet

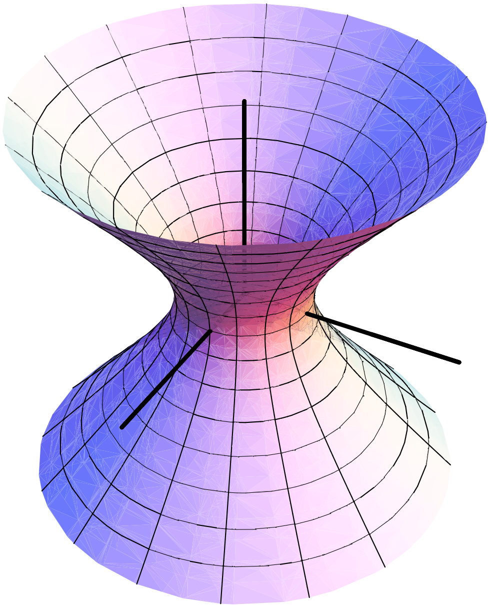

Figure 2. Ruled Hyperboloid

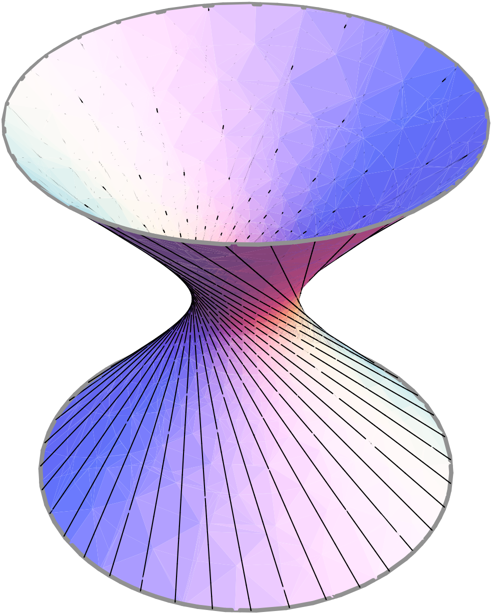

Figure 3. Hexenhut in Natural Position

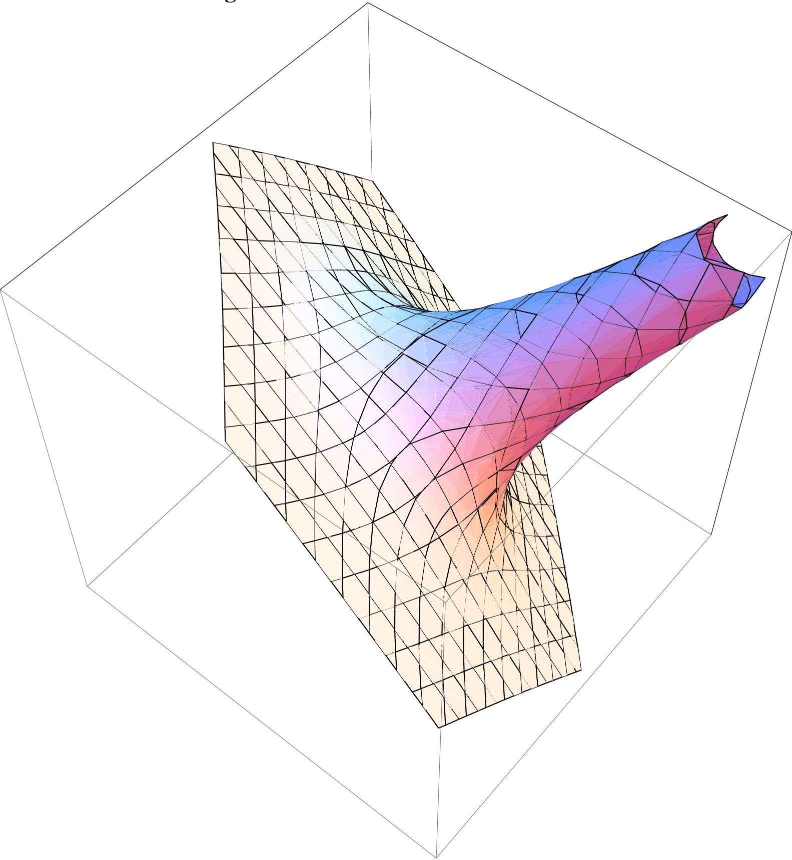

Figure 4. Reoriented Hexenhut

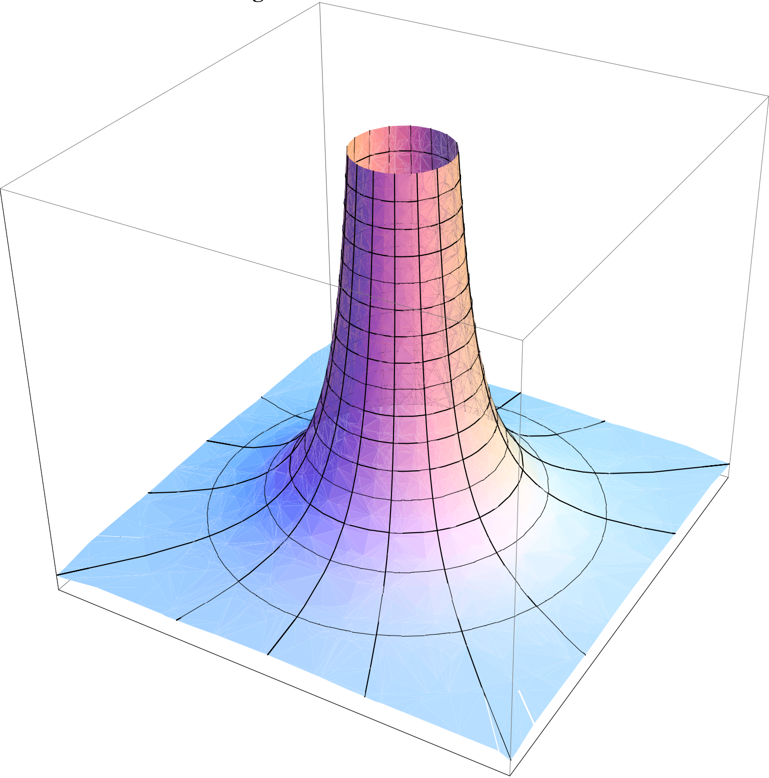

Figure 5. Hexenhut with Inscribed Asymptotic Curves

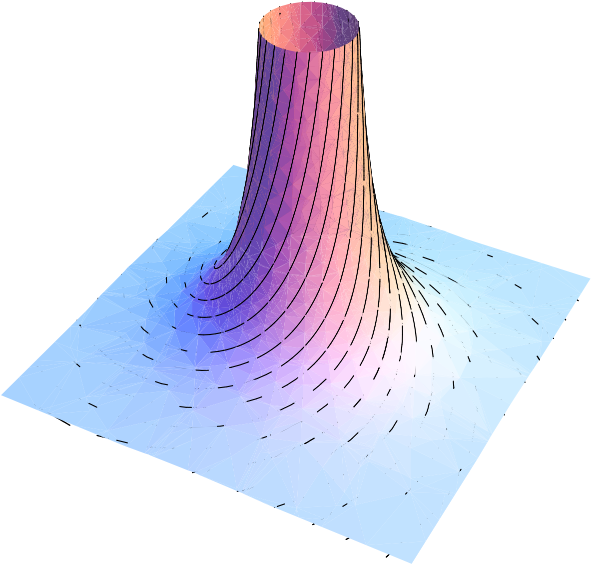

Figure 6. Complete Pseudosphere

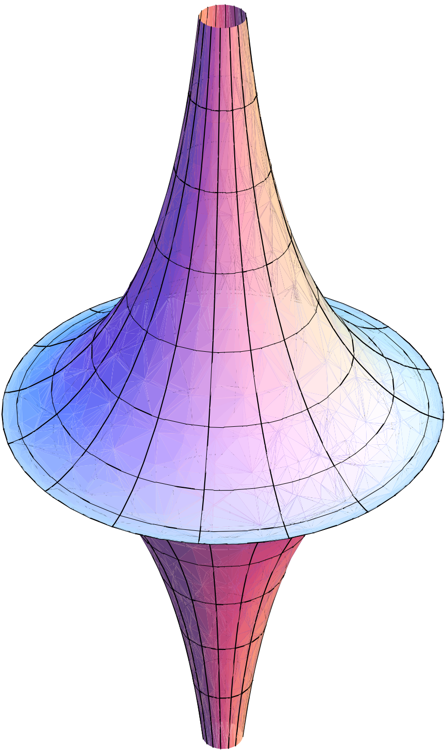

Figure 7. Upper Hemisphere of a Pseudosphere

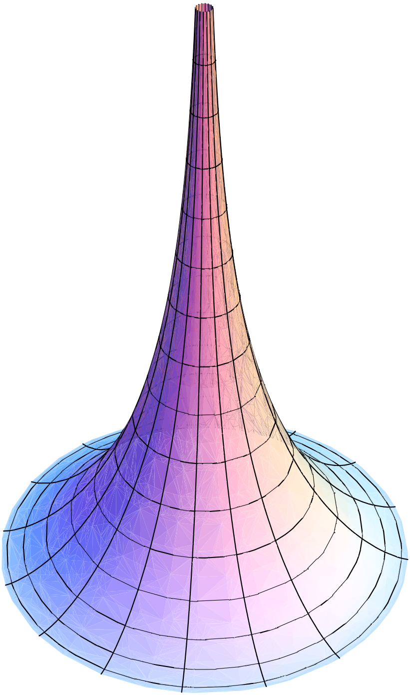

Figure 8. Pseudosphere with Inscribed Asymptotic Curves